# CCRT: Categorical and Combinatorial Representation Theory.

## From combinatorics of universal problems to usual applications.

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Collaboration at various stages of the work and in the framework of the Project *Evolution Equations in Combinatorics and Physics* :

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CIP seminar,

Friday conversations:

For this seminar, please have a look at Slide CCRT[n] & ff.

# Outline

# Goal of this series of talks

The goal of these talks is threefold

1. Category theory aimed at "free formulas" and their combinatorics
2. How to construct free objects
    1. w.r.t. a functor with - at least - two combinatorial applications:
        1. the two routes to reach the free algebra
        2. alphabets interpolating between commutative and non commutative worlds
    2. without functor: sums, tensor and free products
    3. w.r.t. a diagram: limits
3. Representation theory: Categories of modules, semi-simplicity, isomorphism classes i.e. the framework of Kronecker coefficients.
4. MRS factorisation: A local system of coordinates for Hausdorff groups.

**Disclaimer.** – The contents of these notes are by no means intended to be a complete theory. Rather, they outline the start of a program of work which has still not been carried out.

# CCRT[20] MRS and the outer world I.

## The past gives a hand to the future.

1. Last time, we examined Functional and Topological Questions (i.e. Local domains)
   1. Iterated integrals
   2. NCDE $S' = MS$ with asymptotic condition
   3. The topology of $\mathcal{H}(\Omega)$
2. ... and stated some open problems relative to the tree of holomorphic functions generated (continuity, Baire classes). See attached file (CCRT[19]_v8.pdf) in the seminar's page.
3. Today we will begin to explore how the MRS factorization is linked to classical and modern matters.
4. Some concluding remarks.

# Motivation

**Goal of this talk.** – In all our work we have infinite sums and infinite products. If we want to explain this beautiful subject in conferences to other colleagues (math, physics and computer science), we must make precise, explicit and rigorous

- what is the source.
- what is the target.
- what are the limiting processes involved in these two spaces.

The goal of this talk is to (try to) make all these points crystal clear. Let us start, for example, with the following arrow of commutative algebras

$$(\mathbb{C}\langle X\rangle, ш, 1_{X^*}) \xhookrightarrow{\mathrm{Li}_\bullet} (\mathcal{H}(\Omega), \times, 1_\Omega) \tag{1}$$

# Introduction

1. On the RHS of (1), we have the space of holomorphic functions over the connected open set $\Omega$ ($\Omega \subset \widetilde{B}$ with $B = \mathbb{C} \setminus \{0, 1\}$)
2. In practice $\Omega$ is simply connected (in order that $\frac{d}{dz}$ had a section, i.e. the antiderivative), then $\Omega = \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$ or $\Omega = \widetilde{B}$ are usually considered.
3. On the LHS, we have the algebra of noncommutative polynomials embedded in the shuffle algebra of noncommutative series $(\mathbb{C}\langle\langle X \rangle\rangle, ш, 1_{X^*})$.
4. Beginning by this LHS, we observe that two topologies are usually considered.
    1. The topology of stationary convergence.
    2. The topology of Treves.
5. We have (too quickly, I admit) seen these matter last Friday, it leads to the use of MRS factorization.
6. Today we will pursue our step-by-step spiral route (see slide 18).

# Topology of stationary convergence (TSC)

7. It is the product topology on $\mathbb{C}\langle\langle X\rangle\rangle =_{\mathbf{Set}} \mathbb{C}^{X^*}$, or, if one prefers, that of pointwise convergence on the words, **but $\mathbb{C}$ being endowed with the discrete topology**.
8. This topology is standard in combinatorics and computer science because it does not need $\mathbf{k}$ to be endowed with any topology a priori.
9. It is the mode of convergence of Picards's process and the MRS factorization with $\mathbf{k} = \mathcal{H}(\Omega)$ (a strong one indeed as it implies all others).
10. Let us firstly consider finite alphabets (as $X = \{x_0, x_1\}$, $\mathbf{k}$ arbitrary) and give the quantified criterium for $(S_n)_{n\geq 0}$ to converge to $S$

$$(\forall w \in X^*)(\exists N \geq 0)(\forall n \geq N)(\langle S_n|w\rangle = \langle S|w\rangle) \tag{2}$$

# TSC as defined by a distance

11 **Ex1. –** We suppose $\mathcal{X}$ to be finite and $\mathbf{k}$ an arbitrary ring (it is, in our works $\mathbf{k} = \mathbb{C}$ or $\mathcal{H}(\Omega)$, but the following holds true for any ring whatever its characteristics).

12 1) Prove that, when $\mathcal{X}$ is finite, the topology given by the criterium (2) is exactly defined by the distance (called ultrametric distance)

$$d(S,T) = 2^{-\varpi(S-T)} \tag{3}$$

where, for $R \in \mathbf{k}\langle\langle X\rangle\rangle$, $\varpi(R) = \inf\{|w|\}_{w\in supp(R)} \in \mathbb{N} \cup \{+\infty\}$
2) Give an example showing that the result is no longer true if $\mathcal{X}$ is infinite (**Hint. –** Take $\mathcal{X} = \mathcal{Y}$ and consider the sequence of series $S_n = \sum_{1\leq j\leq n} y_j$ (hence, in particular, $S_0 = 0$), show that $S_n$ does converge to $S = \sum_{j\geq 1} y_j$ wrt (2) but NOT wrt the distance (3)).

13 This topology is also that of *Summable Families* in [16] and [27].

14 We will see later the notion of *Summable Family* within (abelian) groups, and before consider a bit what says Riemann's Series Theorem [28].

# TSC and iterated integrals

15 **Ex2.** – Let $\mathcal{X}$ be an alphabet, $(\mathcal{A}, \partial)$ be a differential ring [6] and $(u_x)_{x\in\mathcal{X}}$ a family in $\mathcal{A}$. We suppose that $\partial$ admits a section $\int \in End_{\mathbb{Z}}(\mathcal{A})$ i.e. such that $\partial \circ \int = Id_{\mathcal{A}}$. We extend these elements to $\mathcal{A}\langle\langle\mathcal{X}\rangle\rangle$ term-by-term by

$$\mathbf{d}(S) := \sum_{w\in\mathcal{X}^*} \partial(\langle S|w\rangle)\, w \ ; \ \int S := \sum_{w\in\mathcal{X}^*} [\int \langle S|w\rangle]\, w \tag{4}$$

With $M = \sum_{x\in\mathcal{X}} u_x\, x$, we define the sequence

$$S_0 = 1_{\mathcal{A}\langle\langle\mathcal{X}\rangle\rangle} \ , \ S_{n+1} = 1_{\mathcal{A}\langle\langle\mathcal{X}\rangle\rangle} + \int M.S_n \tag{5}$$

1) Show that this sequence converges, for the topology defined by (3), to a solution of the system

$$\mathbf{d}(S) = M.S \ ; \ \langle S|1_{\mathcal{X}^*}\rangle = 1_{\mathcal{A}} \tag{6}$$

# TSC and iterated integrals/2

16 **Ex2.** – cont'd

2) Show that the result above (convergence to a solution ) holds true even for arbitrary $M \in \mathcal{A}_+\langle\langle\mathcal{X}\rangle\rangle$.

3) Show that, with $\Omega \subset \mathbb{C}$ open and simply connected, $\mathcal{A} = \mathcal{H}(\Omega)$, $\partial = d/dz$ and, with $z_0 \in \Omega$, $\int(f) = \int_{z_0}^{z} f(s)ds$, we get

$$\lim_{n\to\infty} S_n = \sum_{w\in\mathcal{X}^*} \alpha_{z_0}^{z}(w)\, w \tag{7}$$

where $S_n$ is the sequence defined in (5).

4) (Change of lower bounds) Let $\mathcal{F} = (z_w)_{w\in\mathcal{X}^*}$ be an abitrary family of points of $\Omega$. Show that the recursion

$$\alpha_{\mathcal{F}}^{z}(1_{\mathcal{X}^*}) = 1_{\mathcal{H}(\Omega)},\ \alpha_{\mathcal{F}}^{z}(xv) = \int_{z_v}^{z} u_x(s)\alpha_{\mathcal{F}}^{s}(v)ds\ (x \in \mathcal{X},\ \sqsubseteq \in \mathcal{X}^*) \tag{8}$$

is well defined and that $S = \sum_{w\in\mathcal{X}^*} \alpha_{\mathcal{F}}^{z}(w)\, w$ is a solution of (6).

# TSC and iterated integrals/3

17 **Ex2.** – cont'd
5) Formulate a counterpart of the recursion (8) for iterated integrals in an integro-differential ring $(\mathcal{A}, d, \int)$.

18 **Ex3.** –
Let $M$ be a monoid (in fact we will consider $M = \mathcal{X}^*$ or $M = \mathcal{X}^* \otimes \mathcal{X}^*$, in general $M$ is the set of monomials). We have the usual pairing
$\mathbf{k}^M \otimes \mathbf{k}^{(M)} \to \mathbf{k}$, by

$$\langle S|P\rangle := \sum_{m\in M} \langle S|m\rangle\langle P|m\rangle \tag{9}$$

19 A family of series $(S_i)_{i\in I}$ is said *summable* if, for all $m \in M$, $i \mapsto \langle S_i|m\rangle$ is finitely supported and the limit is, by definition, $S = \sum_{m\in M} \sum_{i\in I} \langle S_i|m\rangle\, m$.

# TSC and metric (abelian) groups

20 **Ex4. –**

Let $(G,+,d)$ be an abelian group endowed with a distance $d$. We say that it is a metric group if the operations $(g,h)\to g+h$ and $g\to -g$ are continuous.

1) Let $\mathcal{X}$ be an alphabet and $\mathbf{k}$ a ring. Prove that $(\mathbf{k}\langle\langle\mathcal{X}\rangle\rangle,+,d)$, where $d$ is the distance (3) is a metric group.

2) In a metric group, a family $(g_i)_{i\in I}$ is said *summable*[a] to $S$ if

$$(\forall\epsilon>0)(\exists F\subset_{finite} I)(F\subset F'\subset_{finite} I\Longrightarrow d\Big(\sum_{j\in F'} g_j, S\Big)<\epsilon)$$

3) Show that, if $\mathcal{X}$ is finite, a family $(S_i)_{i\in I}$ of series is summable if, for all $w\in\mathcal{X}^*$, the map $i\to\langle S_i|w\rangle$ is finitely supported. Show that its sum is then

$$S=\sum_{w\in\mathcal{X}^*}\sum_{i\in I}\langle S_i|w\rangle\, w$$

[a] For summability, have a look there
https://mathoverflow.net/questions/289760
https://mathoverflow.net/questions/318843
http://www.cip.ifi.lmu.de/~grinberg/t/21s/lecs.pdf

# TSC and MRS (double series and linear operators)

21 Now, consider the MRS factorization which is one of our precious jewels.

$$\mathcal{D}_X := \sum_{w \in X^*} w \otimes w = \sum_{w \in X^*} S_w \otimes P_w = \prod_{l \in \mathcal{L}ynX}^{\searrow} \exp(S_l \otimes P_l) \qquad (10)$$

22 It is of the form $A = B = C = D$. What do we have ?

- $A = B$ is a definition.
- $B = C$ is the expression of "Bases in Duality".
- $C = D$ is a factorization in an infinite product.

The minimal (and natural) structure where it can take place is that of topological (we have infinite sums and products) rings (see [4], ch. III §6 section 3 and [7]).

23 To understand (and prove) (10) the ultrametric distance (3) will be sufficient. But first, let's have a slide of motivation.

# Factorisation of the diag. series as a resolution of identity.

## Resolution of identity as an infinite product

We are now in the position of writing the principal factorisation of the diagonal series. In here, series multiply by shuffle on the left and concatenation on the right.

$$\mathcal{D}_X := \sum_{w \in X^*} w \otimes w = \sum_{w \in X^*} S_w \otimes P_w = \prod_{l \in \mathcal{L}ynX}^{\searrow} \exp(S_l \otimes P_l) \tag{11}$$

## Application to factorisation of characters

If we have a shuffle-character $\chi : (\mathbf{k}\langle\mathcal{X}\rangle, ш, 1_{\mathcal{X}^*}) \to \mathcal{A}$, we act on the left

$$\chi = \sum_{w \in X^*} \chi(w) \otimes w = \prod_{l \in \mathcal{L}ynX}^{\searrow} \exp(\chi(S_l) \otimes P_l) \tag{12}$$

But with a conc-character $\chi : (\mathbf{k}\langle\mathcal{X}\rangle, \rfloor\wr\backslash\rfloor, 1_{\mathcal{X}^*}) \to \mathcal{A}$, we act on the right

$$\chi = \sum_{w \in X^*} w \otimes \chi(w) = \prod_{l \in \mathcal{L}ynX}^{\searrow} \exp(S_l \otimes \chi(P_l)) \tag{13}$$

# TSC and MRS/2

21. Identity $B = C$ in (10) is an identity between double series (i.e. the algebra $\mathbf{k}\langle\langle \mathcal{X}^* \otimes \mathcal{X}^* \rangle\rangle$)
22. Each $S \in \mathbf{k}\langle\langle \mathcal{X}^* \otimes \mathcal{X}^* \rangle\rangle$ has an intepretation in terms of operators.
    1. Firstly, we can remark that one can always write $S = \sum_{u \in \mathcal{X}^*} S_u \otimes u$ (existence, summabilitity and unicity is left, as an exercise, to the reader)
    2. More generally, a basis $(Q_i)_{i \in I}$, a basis of $\mathbf{k}\langle X \rangle$ being given, one can write uniquely $S$ as $S = \sum_{i \in I} L_i \otimes Q_i$, where $L_i \in \mathbf{k}\langle\langle \mathcal{X} \rangle\rangle$ (again existence, summabilitity and unicity is left to the reader)
    3. Let us remark, in passing that the expressions $L_i \otimes Q_i$ are by no means ambiguous because, although the arrow
    $$\mathbf{k}\langle\langle \mathcal{X} \rangle\rangle \otimes \mathbf{k}\langle\langle \mathcal{X} \rangle\rangle \to \mathbf{k}\langle\langle \mathcal{X}^* \otimes \mathcal{X}^* \rangle\rangle$$
    is not into in general its restriction to $\mathbf{k}\langle\langle \mathcal{X} \rangle\rangle \otimes \mathbf{k}\langle \mathcal{X} \rangle$ is into.
23. To $S = \sum_{i \in I} L_i \otimes Q_i$ one associates $\Phi(S) \in Hom(\mathbf{k}\langle \mathcal{X} \rangle, \mathbf{k}\langle\langle \mathcal{X} \rangle\rangle)$ defined by
$$\Phi(S)[u] := \sum_{i \in I} \langle L_i | u \rangle Q_i \tag{14}$$

# TSC and MRS/3

24 Now, we have a lemma

## Lemma

*Let $(Q_i)_{i\in I}$ be a basis of $\mathbf{k}\langle\mathcal{X}\rangle$ and $(L_j)_{j\in I}$ be its dual family (defined by $\langle L_j|Q_i\rangle = \delta_{ij}$), then*

$$\Phi(\sum_{i\in I} L_i \otimes Q_i) = Id_{\mathbf{k}\langle\mathcal{X}\rangle} \tag{15}$$

25 To conclude, in view of this lemma, it suffices to remark that

$$\Phi(\sum_{w\in X^*} w \otimes w) = Id_{\mathbf{k}\langle\mathcal{X}\rangle} = \Phi(\sum_{w\in X^*} S_w \otimes P_w) \tag{16}$$

26 Now, remarking that, with $\mathfrak{g} = \mathcal{L}ie_{\mathbf{k}}\langle\mathcal{X}\rangle$, one has $\mathbf{k}\langle\langle\mathcal{X}\rangle\rangle = \mathcal{U}(\mathfrak{g})$, $C = D$ is a particular case of the following theorem.

## Theorem, [13]

Let **k** be a $\mathbb{Q}$-algebra and $\mathfrak{g}$ be a Lie algebra which is free as a **k**-module. Let us fix an ordered basis $B = (b_i)_{i\in I}$ (where the ground set $(I, <)$ is totally ordered) of $\mathfrak{g}$. To construct the associated PBW basis of $\mathcal{U} = \mathcal{U}(\mathfrak{g})$, we use the following multiindex notation. For every $\alpha \in \mathbb{N}^{(I)}$, we set

$$B^\alpha = b_{i_1}^{\alpha(i_1)} \cdots b_{i_n}^{\alpha(i_n)} \in \mathcal{U} \tag{17}$$

where $\{i_1, \cdots, i_n\} \supset \mathrm{supp}(\alpha)$ (and $i_1 < \cdots < i_n$).
Consider the linear coordinate forms $B_\beta \in \mathcal{U}^\vee$ defined by

$$\langle B_\beta | B^\alpha \rangle = \delta_{\alpha,\beta}. \tag{18}$$

We will also use the elementary multiindices $e_i \in \mathbb{N}^{(I)}$ defined for all $i \in I$ by $e_i(j) = \delta_{i,j}$.

# Main theorem/2

## Theorem cont'd

Then:[a]

1. We have

$$B_\alpha \circledast B_\beta = \frac{(\alpha+\beta)\,!}{\alpha\,!\,\beta\,!} B_{\alpha+\beta} \tag{19}$$

   and

$$B_{\alpha(i_1)e_{i_1}+\cdots+\alpha(i_k)e_{i_k}} = \frac{B_{e_{i_1}}^{\circledast\alpha(i_1)} \circledast \cdots \circledast B_{e_{i_k}}^{\circledast\alpha(i_k)}}{\alpha(i_1)\,!\cdots\alpha(i_k)\,!}. \tag{20}$$

2. The following infinite product identity holds:

$$Id_{\mathcal{U}} = \circledast_{i\in I}^{\rightarrow}\; e_{\circledast}^{Im(B_{e_i}\otimes B^{e_i})} = \overrightarrow{\prod_{i\in I}}\; e_{\circledast}^{Im(B_{e_i}\otimes B^{e_i})} \tag{21}$$

   within $End(\mathcal{U})$.

[a] We use the notation $\alpha!$ for $\alpha \in \mathbb{N}^{(I)}$; this is the product $\alpha! = \prod_{i\in I}\alpha_i!$.

# Concluding remarks

1. Motivation
2. Introduction
3. Topology of stationary convergence (TSC)
4. TSC as defined by a distance
5. TSC and iterated integrals
6. TSC and metric (abelian) groups
7. TSC and MRS/2
8. Factorisation of the diag. series as a resolution of identity.
9. TSC and MRS/2 (double series and linear operators)
10. Main theorem

THANK YOU FOR YOUR ATTENTION !

By the way, below the bibliography cited and some more running titles.

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